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Quasi-analytical Method for Solving Nonlinear Differential Equations for Turbulent Self-Confined Magneto-Plasmas

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A quasi-analytical method is presented which permits calculating highly nonlinear behavior of selfconfined plasma configurations. Using the possibilities of the analytic computer code REDUCE, the equations are transformed to $(\mathbf{k}, \omega)$-Fourier space. The use of analytical methods also in back-transformation avoids difficulties with steep gradients. A dispersion relation is established from which growth rates of instabilities can be calculated. The application of the method is demonstrated for Burgers' equation with a specific initial condition for which an analytical solution is known.

1. INTRODUCTION

Until recently, problems of magneto-plasma flows were generally calculated with numerical methods, using difference schemes to solve the system of nonlinear differential equations. For such magneto-plasma flows, the magnetic field generally has a simple configuration, while the flow equations can rarely be simplified. Strongly nonlinear behavior and turbulence phenomena growing out of large amplitude instabilities play an important role, also various transport processes have to be taken into account. This macro-turbulent regime is frequently characterized by large Reynolds numbers, where values of 10^9 and larger are not uncommon in turbulent structures. This requires solution of the complete nonlinear equations over time intervals that are large compared to the Alfvèn time. However, the common numerical schemes used in hydrodynamics and gas dynamics fail in many cases due to several reasons. First, the size of the turbulent structures is essentially determined by the magnitude of the grid chosen and, second, the mass conservation law is often violated. The problems encountered in the numerical treatment of tur-

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bulence are discussed in detail by Saffman [1]. Last but not least, most numerical schemes require the introduction of the von Neumann viscosity to avoid numerical instability for otherwise steep gradients, resulting in erroneous values for gradient-driven instabilities and transport processes.

From previous experience with the calculation of turbulent flows in hydrodynamics, the first two authors developed a REDUCE code [2, 3, 4] for the manipulation of strongly nonlinear equations, combining it with FORTRAN for numerical evaluation. This was then applied to the consideration of extremely strong turbulence plasmas [5], where essentially the strong nonlinear terms were retained while linear effects were neglected.

Besides posing difficulties in physical interpretation, this procedure is unsuitable for describing the evolution of turbulence from originally stable plasma states via instabilities. Particularly, the calculation of plasma focus dynamics with the turbulent late phase after $m = 0$ instability [6] requires a method permitting a description of the time-dependent development of turbulence. For this purpose, one of us (H.J.K.) decided to treat the problem by a transformation into Fourier space, similarly as was previously done. e.g., in an analytical theory of pinch instabilities [7]. Contrary to the linearization carried out in this paper by Wilhelm, the complete nonlinear terms were to be retained in this new method. Furthermore, the transformation into $(\mathbf{k}, \omega)$-space and establishing of the macroscopic dispersion relation were to be handled by REDUCE. Also back-transformation was to be made at least in part analytically.

The REDUCE–FORTRAN hybrid code, developed with the above considerations in mind, handles the derivation of the dispersion relation analytically, while the solution of the dispersion relation is done numerically. This numerical solution is then fitted to an analytical formula for further symbolic manipulation. The interchange of symbolic manipulation, numerical calculation, and analytical fit programs is characteristic for this hybrid code. Due to the fact that also back-transformation from $(\mathbf{k}, \omega)$- into $(\mathbf{r}, t)$-space is carried out analytically, the problems connected with steep gradients do not arise.

This new method was essentially developed for plasma flows in magnetic fields of simple configuration. It is applicable to a number of problems for self-confined plasmas where the radial dimension extends to infinity, treated hitherto by various well-known MHD-codes. However, it is feasible to apply it only in those cases, where the magnetic field configuration is relatively simple and the emphasis is on studying unstable plasma behavior. This method was first applied to the nonlinear development of the $m = 0$ instability in the plasma focus [8] using a one-fluid formalism and then extended to further phases of plasma focus operation employing a two-fluid theory [9]. At present it is applied to a four-fluid description of magneto-plasma dynamics [10] including also microturbulence, describing plasma focus and z-pinch phenomena.

This paper describes the mathematical procedure used in solving the systems of nonlinear differential equations and does not contain details of programming in REDUCE. For details concerning REDUCE, see references [2–4]. The programm-

ing procedure for the REDUCE–FORTRAN hybrid code will be published subsequently. A paper containing results from calculating the dynamic phenomena in the POSEIDON plasma focus experiment is in preparation and will be published elsewhere.

In the following, the calculation procedure is explained in detail and demonstrated for the simple case of Burgers' equation with a delta-function for the initial conditions. The reasons for using Burgers' equation with this initial condition are:

(1) It represents the type of nonlinear equation considered for solution, cf. Eq. (1),

(2) it is Fourier-transformable,

(3) it has an analytical solution against which the accuracy of the proposed method of solution can be checked. The use of Burgers equation does not imply a general applicability to all nonlinear Navier–Stokes problems with the present state of this method. An extension to bounded inhomogeneous systems is in preparation.

As a further example, the solution of Burgers' equation with $F(x) = U \sin kx$ for the initial distribution is given.

2. Method of Solution

As the method of solution described in this section shall mainly be applied to problems of magneto-plasma flows, the system of nonlinear differential equations to be solved consists of the laws of conservation of mass, momentum and energy, together with the various transport coefficients, and the Maxwell equations, as given in Ref. [10]. The types of equations in this reference can be written in a general form, Eq. (1). Thus it is required to solve a system of nonlinear equations of pth order representing a general form of the Navier–Stokes type of partial differential equations,

$$D^{\mathrm{I}}_{\mu\nu}\left(\nabla, \frac{\partial}{\partial t}\right)\psi_\nu(\mathbf{r}, t) + \psi_\lambda(\mathbf{r}, t)\, D^{\mathrm{II}}_{\lambda\mu\nu}\left(\nabla, \frac{\partial}{\partial t}\right)\psi_\nu(\mathbf{r}, t) = F_\mu(\mathbf{r}, t). \tag{1}$$

$D^{\mathrm{I}}_{\mu\nu}(\nabla, \partial/\partial t)$ is the operator describing the linear terms, while $D^{\mathrm{II}}_{\lambda\mu\nu}(\nabla, \partial/\partial t)$ contains all nonliner effects. $F_\mu(\mathbf{r}, t)$ represents the inhomogeneous part of Eq. (1). The Einstein summation convention for the subscripts ($\nu, \lambda = 1,..., p$) is adhered to. To achieve a solution we have to execute the following steps:

(i) *Programming of the system of differential equations* (1) in vector form, where every component of a vector equation is represented by a value of the subscript μ of Eq. (1). This step will be executed by REDUCE.

(ii) *Separation of the dynamical variables* into a constant unperturbed value and "perturbation" series by a "perturbation" ansatz,

$$\psi_\mu(\mathbf{r}, t) = \psi_{\mu 0} + \tilde{\psi}_\mu(\mathbf{r}, t), \tag{2}$$

$$\tilde{\psi}_\mu(\mathbf{r}, t) = \sum_{\kappa=1}^{M} \varepsilon^\kappa \tilde{\psi}_{\mu\kappa}(\mathbf{r}, t). \tag{3}$$

In Eq. (2), $\psi_{\mu 0}$ is a constant representing an equilibrium solution of Eq. (1). This means that assuming

$$\tilde{\psi}_{\mu\kappa}(\mathbf{r}, t) \to 0 \qquad \text{uniformly for } |\mathbf{r}| \to \infty \quad \text{or} \quad |t| \to \infty$$

there follows

$$\psi_\mu(\mathbf{r}, t) \to \psi_{\mu 0} \qquad (|\mathbf{r}| \to \infty \text{ or } |t| \to \infty).$$

Since every derivation in $\mathbf{r}$ or t will vanish at infinity, $\psi_{\mu 0}$ is the "unperturbed" solution of (1). $\psi_{\mu 0}$ must not be interpreted as an initial value of the dynamical variable $\tilde{\psi}_{\mu\kappa}(\mathbf{r}, t)$, as the initial value is also determined by setting $t = t_0$ in the expansion (3). The complete spatial and temporal variations of $\psi_\mu(\mathbf{r}, t)$ are described by the magnitude $\tilde{\psi}_\mu(\mathbf{r}, t)$. For further details see step (iii).

The operations described by Eqs. (2) and (3) are carried out by REDUCE. ε is an ordering parameter solely introduced for programming convenience. With the aid of this ordering parameter it is later possible to sort the equations in powers of ε with the aid of REDUCE (see step (iii)). The value of ε thus should be unity and the convergence of the series (3) is solely determined by the behavior of the $\tilde{\psi}_{\mu\kappa}(\mathbf{r}, t)$. The variable $\tilde{\psi}_{\mu\kappa}(\mathbf{r}, t)$ is defined by Eq. (6). Thus, Eqs. (2) and (3) cannot in general be interpreted as a conventional perturbation ansatz.

It is possible, unlike the behavior of the usual perturbation series, that up to a value M_0 of the summation index κ in (3), the solutions $\tilde{\psi}_{\mu\kappa}(\mathbf{r}, t)$ show growing tendency

$$\tilde{\psi}_{\mu\kappa}(\mathbf{r}, t) > \tilde{\psi}_{\mu\kappa-1}(\mathbf{r}, t), \qquad \kappa < M_0, \tag{4}$$

so that convergence of the series requires expansion to high orders of κ ($\kappa \gg M_0$). In the cases calculated by us previously, we had to develop this series up to an order of $\kappa \approx 100$ to 200.

The convergence of (3) cannot be proven for the general case of Eq. (1). Thus, the following procedure for ascertaining the convergence of the series (3) is recommended:

(a) For every specific problem the $\tilde{\psi}_{\mu\kappa}(\mathbf{r}, t)$ must be calculated up to order M (see step (iii) ff).

(b) These solutions are summed up according to Eq. (3). If it is ascertained in a calculation of members of order higher than M (e.g., up to $M + 20$) that the result deviates only within an accepted range of accuracy, then the series is terminated.

(c) For an alternating series it will have to be investigated how large the difference of the numbers turns out to be in relation to their magnitude. If this quotient is smaller than an acceptable limit, there follows another termination condition for the series. In the cases calculated by the authors, the convergence was also ascertained using Leibniz's convergence criterion. The $\tilde{\psi}_{\mu\kappa}(\mathbf{r}, t)$ in these cases showed, as a rule, alternating behavior. Hence, it is possible that in summing Eq. (3), differences of two large terms occur. This does not play a negative role in the analytical part of the code (REDUCE). However, in the numerical part (FORTRAN), care must be taken that no serious errors are introduced by such differences of large terms which may lead to numerical instabilities.

The method discussed here shows similarity to the method of weighted residuals or collocation method, e.g., that of Kantorovich–Ritz [11].

(iii) *Inserting the ansatz* (3) into the system of differential equations (1) and ordering to powers of ε.

Performing this with REDUCE, there result the following expressions from Eq. (1), in zeroth order of ε,

$$(D^{\mathrm{I}}_{\mu\nu} + \psi_{\lambda 0} D^{\mathrm{II}}_{\nu\lambda\mu})\, \psi_{\nu 0} = 0, \tag{5}$$

in κth order of ε,

$$(D^{\mathrm{I}}_{\mu\nu} + \psi_{\lambda 0} D^{\mathrm{II}}_{\nu\lambda\mu})\, \tilde{\psi}_{\nu\kappa} = R_{\mu\kappa}, \tag{6}$$

with

$$(\kappa = 1): \qquad R_{\mu 1}(\mathbf{r}, t) = F_{\mu}(\mathbf{r}, t), \tag{7a}$$

$$(\kappa > 1): \qquad R_{\mu\kappa}(\mathbf{r}, t) = -\sum_{j=1}^{\kappa-1} [\tilde{\psi}_{\lambda j} D^{\mathrm{II}}_{\lambda\mu\nu} \tilde{\psi}_{\nu,\kappa-j}]. \tag{7b}$$

In connection with Eq. (5), the following should be noted. If the sum (3) would have included $\varepsilon = 0$, thus,

$$\tilde{\psi}_{\mu}(\mathbf{r}, t) = \sum_{\kappa=0}^{M} \varepsilon^{\kappa} \tilde{\psi}_{\mu\kappa}(\mathbf{r}, t),$$

then the zeroth member would have been

$$\varepsilon^0 \tilde{\psi}_{\mu 0}(\mathbf{r}, t) = \tilde{\psi}_{\mu 0}(\mathbf{r}, t)$$

with the $\psi_{\mu 0}$ a function of $\mathbf{r}$ and t. As, however, $\psi_{\mu 0}$ is a constant; $\tilde{\psi}_{\mu 0}(\mathbf{r}, t)$ must vanish. This is achieved by starting the series with $\kappa = 1$. Nevertheless, Eq. (5) describing the $\psi_{\mu 0}$ represents the zeroth order of ε.

Equations (6) represent now a system of linear differential equations in every order $\kappa \geqslant 1$ of ε with inhomogeneous parts $R_{\mu\kappa}$ expressed by Eqs. (7). This system

describes the excitation of the $\tilde{\psi}_{\nu\kappa}$ arising from the solutions $\tilde{\psi}_{\nu\kappa'}$ $(\kappa' < \kappa)$. Thus, in every order of ε, the linear problem must be solved successively and the addition of all solutions $\tilde{\psi}_{\mu\kappa}$ yields the total solution $\tilde{\psi}_{\mu}(\mathbf{r}, t)$ up to the order κ.

It is now clear how nonlinearity develops. Contrary to the usual perturbation ansatz, the excitation of the κth term can be stronger than that of the $(\kappa-1)$th term if the damping of the amplitudes included in the $D^{\mathrm{I}}_{\mu\nu}$-term can be neglected with respect to the advective $D^{\mathrm{II}}_{\lambda\mu\nu}$-term. Thus, in every order, the "perturbation" will grow due to the magnitude of the "inhomogeneous" term until a change of phase of the right-side-term of Eq. (6) gives rise to a decremental behavior of the solutions, thus enforcing the convergence of the system.

With the abbreviation

$$D_{\mu\nu}\left(\nabla, \frac{\partial}{\partial t}\right) = D^{\mathrm{I}}_{\mu\nu}\left(\nabla, \frac{\partial}{\partial t}\right) + D^{\mathrm{II}}_{\lambda\mu\nu}\left(\nabla, \frac{\partial}{\partial t}\right) \cdot \psi_{\lambda 0}, \tag{8}$$

we can now write Eq. (6) as

$$\mathbf{D}\left(\nabla, \frac{\partial}{\partial t}\right) \tilde{\boldsymbol{\psi}}_{\kappa}(\mathbf{r}, t) = \mathbf{R}_{\kappa}(\mathbf{r}, t), \tag{9}$$

where $\mathbf{D}(\nabla, \partial/\partial t)$ is a linear dispersion matrix of order $p \times p$,

$$\mathbf{D}\left(\nabla, \frac{\partial}{\partial t}\right) = \left(\left(D_{\mu\nu}\left(\nabla, \frac{\partial}{\partial t}\right)\right)\right), \tag{10a}$$

and $\tilde{\boldsymbol{\psi}}_{\kappa}(\mathbf{r}, t)$ and $\mathbf{R}_{\kappa}(\mathbf{r}, t)$ are column vectors of order p,

$$\tilde{\boldsymbol{\psi}}_{\kappa}(\mathbf{r}, t) = \begin{vmatrix} \psi_{1\kappa}(\mathbf{r}, t) \\ \vdots \end{vmatrix}, \qquad \mathbf{R}_{\kappa}(\mathbf{r}, t) = \begin{vmatrix} R_{1\kappa}(\mathbf{r}, t) \\ \vdots \end{vmatrix}. \tag{10b}$$

The general solution of (9) is given by

$$\tilde{\boldsymbol{\psi}}_{\kappa}(\mathbf{r}, t) = \tilde{\boldsymbol{\psi}}_{h\kappa}(\mathbf{r}, t) + \tilde{\boldsymbol{\psi}}_{i\kappa}(\mathbf{r}, t), \tag{11a}$$

where $\tilde{\boldsymbol{\psi}}_{h\kappa}$ is the homogeneous term of the solution,

$$\mathbf{D}\left(\nabla, \frac{\partial}{\partial t}\right) \tilde{\boldsymbol{\psi}}_{h\kappa}(\mathbf{r}, t) = 0 \tag{11b}$$

and $\tilde{\boldsymbol{\psi}}_{i\kappa}$ the inhomogeneous part

$$\tilde{\boldsymbol{\psi}}_{i\kappa}(\mathbf{r}, t) = \int d^3r'\, dt'\, \mathbf{G}_{\kappa}(\mathbf{r}, \mathbf{r}'; t - t')\, \mathbf{R}_{\kappa}(\mathbf{r}', \mathbf{t}'), \tag{11c}$$

with $\mathbf{G}_{\kappa}(r, r'; t - t')$ being the matrix of Green's function of the system.

As this proposed method of calculation shall be applied to the treatment of systems of differential equations (1) of the Navier–Stokes type which always con-

tain a viscous term in the operator $D^{\mathrm{I}}_{\mu\nu}$, the variables $\tilde{\psi}_\mu(\mathbf{r}, t)$ vanish sufficiently fast for $r \to \infty$ and/or $t \to \infty$, so that their square-integrability is secured. Again assuming that the series (3) converges, the $\tilde{\psi}_{\mu\kappa}(\mathbf{r}, t)$ defined by Eq. (6) are also square-integrable. Thus the definition in Eq. (11c) for the inhomogeneous part of the solution is justified. Since the form of the dispersion matrix **D** does not depend on the order κ, we only have to solve the homogeneous problem in first order of ε ($\kappa = 1$). For higher orders it is sufficient to solve the inhomogeneous part of Eq. (9).

For further treatment of the integral in Eq. (11) the following remarks are necessary.

The purpose of the calculation method is the description of the dynamics of self-confined plasmas, where the radial dimension extends to infinity. Thus there are no boundary surfaces and Green's function in Eq. (11c) can only depend on $\mathbf{r} - \mathbf{r}'$. Therefore Eq. (11c) becomes a convolution integral

$$\tilde{\boldsymbol{\psi}}_{i\kappa}(\mathbf{r}, t) = \int d^3r'\, dt'\, \mathbf{G}_\kappa(\mathbf{r} - \mathbf{r}', t - t')\, \mathbf{R}_\kappa(\mathbf{r}', t')$$

and the following procedures are justified.

Of course, if there are boundary surfaces, the description of $\mathbf{G}_\kappa(\mathbf{r}, \mathbf{r}'; t - t')$ as a retarded Green's function fails. The treatment of this case needs a modification of the method discussed in this paper. This extension is in progress at present and will be published in further papers.

(iv) *Wave ansatz for the* $\tilde{\psi}_\mu(\mathbf{r}, t)$.

The considerations for the selection of the $\tilde{\boldsymbol{\psi}}_\kappa(\mathbf{r}, t)$ were already discussed in step (iii). As this method is at the present time essentially applied to the solution of the turbulent magneto-plasma dynamic equations and the turbulent state grows out of instabilities and nonlinear wave development, as is well known, the emphasis is on a description in Fourier space. Hence we select the Fourier ansatz,

$$\tilde{\boldsymbol{\psi}}_{j\kappa}(\mathbf{r}, t) = \int \frac{d^3k\, d\omega}{(2\pi)^4}\, \hat{\boldsymbol{\psi}}_{j\kappa}(\mathbf{k}, \omega)\, e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \tag{12}$$

which is substituted into the equations by REDUCE. In Eq. (12), **k** is the wave vector, ω the angular frequency and $j = h, i$. This can be done for all realistic problems, where the square-integrability of the $\tilde{\psi}_{\nu\kappa}(\mathbf{r}, t)$ is given. It is important to emphasize the fact that the Fourier integral theorem is used and not an expansion in Fourier series. Therefore the entire $(\mathbf{k}, \omega)$-space gives contributions to the Fourier transform of $\tilde{\psi}_{\nu\kappa}(\mathbf{r}, t)$ and there is no error usually due to cutting off the Fourier-series. Inserting Eqs. (12) and (11a) into Eq. (9), and considering the relationships of differential operators in $(\mathbf{r}, t)$-space with wave number and angular frequency in Fourier space,

$$\frac{\partial}{\partial t} \to i\omega, \qquad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \to -i\mathbf{k} = -i(k_x, k_y, k_z), \tag{13}$$

we obtain, by using the folding theorem,

$$\mathbf{D}(-i\mathbf{k}, i\omega)\,\hat{\psi}_{h\kappa}(\mathbf{k}, \omega) + \mathbf{D}(-i\mathbf{k}, i\omega)\,\underbrace{\mathbf{G}_{\kappa}(\mathbf{k}, \omega)\,\hat{\mathbf{R}}_{\kappa}(\mathbf{k}, \omega)}_{\hat{\psi}_{i\kappa}(\mathbf{k}, \omega)} = \hat{\mathbf{R}}_{\kappa}(\mathbf{k}, \omega). \tag{14}$$

Instead of a differential equation system we now have a linear system of algebraic equations in $(\mathbf{k}, \omega)$-space, which we can solve by algebraic manipulations.

(v) *Linear dispersion relation and solution in* $(\mathbf{k}, \omega)$*-space.*

To solve the homogeneous problem

$$\mathbf{D}(-i\mathbf{k}, i\omega)\,\hat{\psi}_{h\kappa}(\mathbf{k}, \omega) = 0 \tag{15}$$

we have to formulate the linear dispersion relation

$$\det(\mathbf{D}(-i\mathbf{k}, i\omega)) = 0. \tag{16}$$

Computing this with REDUCE, there generally follows an implicit equation

$$D(\mathbf{k}, \omega) = 0. \tag{17}$$

Thus the solution yields a relationship between the dependence of the frequency and that of the $\mathbf{k}$-vector leading to a relationship between the spatial and the time-dependent development of the solution by Fourier transformation.

It is now essential to solve the dispersion relation. The first step is to determine its zeros. The Nyquist plot [12] permits a graphical determination of the zeros of Eq. (17). In addition Nyquist has shown that, also for the nonlinear case, the growth and decay rates of all instabilities of the system are determined by these singularities. A suitable numerical method for the determination of the zeros of Eq. (17) was developed by Räuchle and collaborators [13, 14, 15]. It is pointed out that the discussion here only concerns the determination of the zeros and the numerical solution of the dispersion relation. The question whether a resulting instability is absolute or convective will have to be investigated separately if such information should be required [16].

From the implicit formula (17) we now derive the explicit $\mathbf{k}$-dependence of ω,

$$D(\mathbf{k}, \omega) = 0 \leftrightarrow \omega = \omega_q(\mathbf{k}) \qquad (q = 1,..., Q \leqslant p). \tag{18}$$

This cannot be done generally in a strictly analytical procedure by REDUCE. Hence, we have to employ numerical methods (REDUCE–FORTRAN-hybrid code) to find the explicit $\mathbf{k}$-dependence of ω. This is not a critical point, however, since we can fit these results by available numerical routines within the range of the parameters obtained from experiment, thus obtaining an analytical formula for $\omega(\mathbf{k})$. The imaginary part of $\omega_q(\mathbf{k})$,

$$\gamma(\mathbf{k}) = \mathrm{Im}(\omega_q(\mathbf{k})) \qquad (q = 1,..., Q \leqslant p), \tag{19}$$

yields the linear growth or decay rates of the instabilities. Therefore, we can write for the homogeneous solution in $(\mathbf{k}, \omega)$-space,

$$\hat{\psi}_{h\kappa}(\mathbf{k}, \omega) = 2\pi \sum_{q=1}^{Q} \hat{\psi}_h(\mathbf{k}, \omega_q(\mathbf{k}))\, \delta(\omega - \omega_q(\mathbf{k})). \tag{20}$$

To solve the inhomogeneous part of the system (14), we have to calculate Green's function $\hat{\mathbf{G}}_\kappa(\mathbf{k}, \omega)$. It is immediately seen from (14) that

$$\hat{\mathbf{G}}_\kappa(\mathbf{k}, \omega) = \mathbf{D}^{-1}(-i\mathbf{k}, i\omega) = \frac{\mathbf{D}^+(-i\mathbf{k}, i\omega)}{\det(\mathbf{D}(-i\mathbf{k}, i\omega))} = \frac{\mathbf{D}^+(-i\mathbf{k}, i\omega)}{D(\mathbf{k}, \omega)}, \tag{21}$$

with $\mathbf{D}^+$ being the adjoint matrix of $\mathbf{D}$ for all orders of ε. Summarizing, we obain the following solutions for the system in $(\mathbf{k}, \omega)$-space from (20) and (21),

$$\kappa = 1: \qquad \hat{\psi}_1(\mathbf{k}, \omega) = \sum_{q=1}^{Q} \hat{\psi}_h(\mathbf{k}, \omega_q(\mathbf{k}))\, \delta(\omega - \omega_q(\mathbf{k})) + \frac{\mathbf{D}^+(-i\mathbf{k}, i\omega)}{D(\mathbf{k}, \omega)} \hat{\mathbf{R}}_1(\mathbf{k}, \omega), \tag{22}$$

$$\kappa > 1: \qquad \hat{\psi}_\kappa(\mathbf{k}, \omega) = \frac{\mathbf{D}^+(-i\mathbf{k}, i\omega)}{D(\mathbf{k}, \omega)} \hat{\mathbf{R}}_\kappa(\mathbf{k}, \omega). \tag{23}$$

For the solution of the homogeneous part of the dispersion relation, the introduction of boundary values may be required. For details, see step (viii).

(vi) *Elimination of ω by complex integration.*

Since the $(\mathbf{r}, t)$-dependence of the solutions is required, we have to execute the retransformation of the $\hat{\psi}_\kappa(\mathbf{k}, \omega)$ from Eqs. (22) and (23) according to Eq. (12). The ω-integration is performed first. The first term of (22) is easily integrated using the properties of the δ-function. To calculate the inhomogeneous term of (22) and Eq. (23), it is practical to carry out the ω-integration for Green's function (21) and then use the folding integral (11c) for evaluating the time-dependent solution. We have to evaluate the integral

$$\mathbf{g}(\mathbf{k}, t - t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\mathbf{D}^+(-i\mathbf{k}, i\omega)}{D(\mathbf{k}, \omega)} e^{i\omega(t-t')}. \tag{24}$$

If we close the contour in the upper complex ω half plane by a semicircle at infinity (C_2), all poles of $\hat{\mathbf{G}}(\mathbf{k}, \omega)$ situated in the upper half plane will be enclosed as shown in Fig. 1 with the pole ω_l.

In general, the path of integration is not straightforward. In most cases, the path of integration follows along C_1 and encircles the singular pole as was shown by Landau [17].

However, in a few cases a simpler choice for the path of integration is possible, such as shown in Fig. 1, due to the special time behavior of Green's function. The contribution of C_2 will vanish if $t - t' > 0$ since $\mathbf{D}^+$ and $D(\omega, \mathbf{k})$ only contain

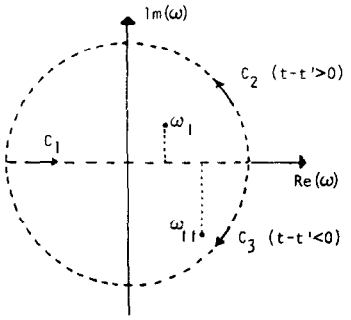

FIGURE 1

powers of ω and $e^{-i\omega(t-t')} \to 0$ for $\omega \to i\infty$. If we close the contour in the lower half plane (C_3), enclosing now all poles of $\hat{\mathbf{G}}(\mathbf{k}, \omega)$ in the lower half plane (ω_{II} in Fig. 1), the contribution of C_3 will vanish in the same way if $t - t' < 0$. Thus the value of (24) will be determined by the contribution of the poles which are the zeros of the dispersion relation $D(\omega, \mathbf{k})$.

Complex integration yields from (24), see also [15],

$$\mathbf{g}(\mathbf{k}, t-t') = \sum_{q_u} \hat{\mathbf{g}}_{qu}(\mathbf{k}) e^{i\omega q_u^{(\mathbf{k})}(t-t')} \Theta(t-t') - \sum_{q_l} \hat{\mathbf{g}}_{ql}(\mathbf{k}) e^{i\omega q_l^{(\mathbf{k})}(t-t')} \Theta(t'-t), \tag{25}$$

with

$$\hat{\mathbf{g}}_{qi}(\mathbf{k}) = i \cdot Res(\hat{\mathbf{G}}(\mathbf{k}, \omega))_{\omega = \omega_{qi}(\mathbf{k})},$$

where q_u and q_l represent the number of poles in the upper and lower half-plane, respectively, and $\Theta(t-t')$ is the Heaviside function. This can be inserted in Eq. (11c) and we obtain, by setting

$$\mathbf{R}_\kappa(\mathbf{r}', t') = \int \frac{d^3k'}{8\pi^3} \tilde{\mathbf{R}}_\kappa(\mathbf{k}', t')\, e^{-i\mathbf{k}' \cdot \mathbf{r}'} \tag{26a}$$

and

$$\int d^3r' e^{-i\mathbf{r}'(\mathbf{k}-\mathbf{k}')} = 8\pi^3 \delta(\mathbf{k}-\mathbf{k}') \tag{26b}$$

the following result for the inhomogeneous part of the system,

$$\tilde{\psi}_{i\kappa}(\mathbf{r}, t) = \sum_{q_u} \int \frac{d^3k}{8\pi^3} \hat{\mathbf{g}}_{qu}(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{r}} \int dt'\, e^{i\omega_{qu}(\mathbf{k})(t-t')} \tilde{\mathbf{R}}_\kappa(\mathbf{k}, t') - \sum_{q_l} \int \frac{d^3k}{8\pi^3} \hat{\mathbf{g}}_{ql}(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{r}} \int dt\, e^{i\omega_{ql}(\mathbf{k})(t-t')} \tilde{\mathbf{R}}_\kappa(\mathbf{k}, t'). \tag{27}$$

Finally we have to evaluate the time-dependent integral in Eq. (27). The method of integration depends on the specific problem and cannot be performed generally. There are various possibilities, e.g., integration in closed form, or by partial integration, or by evaluating the integrand in a power series of t', which can be realized by analytic (REDUCE) manipulations or by numerical treatment. Thus we obtain for each κ the complete solution $\tilde{\psi}_{\nu\kappa}(\mathbf{r}, t)$ as an integral over $\mathbf{k}$-space containing the Fourier transforms of all solutions $\tilde{\psi}_{\nu\kappa'}(\mathbf{r}, t)$ with $\kappa' < \kappa$. Hence we are able to compute all solutions as an integral of $\mathbf{k}$-space successively up to the order desired.

(vii) *Integration over* $\mathbf{k}$*-space and summing up the solutions.*

To determine the local variation of $\tilde{\psi}_{\nu\kappa}(\mathbf{r}, t)$ we have to perform the $\mathbf{k}$-integration of Eq. (27). There are few possibilities of doing this in a completely analytical treatment.

As Green's function or the linear dispersion relation, respectively, are the same in every order, the solution in every order can formally be written as a multiple folding integral in which only the general solutions of first order remain. As is discussed in the following step (viii), initial and boundary conditions are also solved by their satisfying the first order solution. With the complete solution of the problem in first order thus the solutions of every order can be calculated by successive integration over $\mathbf{k}$, the complete solution in all orders being obtained by summation. This integration can in general be carried out only by numerical procedures.

A more tractable method proposed here is the alternative use of fit programs. Here, the result of the numerical solution for $\tilde{\Psi}_{i2}(\mathbf{r}, t)$ from Eq. (27) is fitted to an analytical expression. With this, the excitation term $\tilde{\mathbf{R}}_3(\mathbf{k}, t)$ for the calculation of the solutions of third order $\tilde{\Psi}_{i3}(\mathbf{r}, t)$ from Eq. (27) can then be formulated analytically, the third order solution itself is again obtained numerically and then again fitted to an analytical expression. With this, the fourth order solution is started, etc. This yields a procedure where exactly the same REDUCE manipulation can be employed in every order and only the FORTRAN part in the hybrid code is calculated anew.

In some cases, where, e.g., the variable $\psi_\mu(\mathbf{r}, t)$ is almost constant everywhere with the exception of strong local gradients in $\mathbf{r}$-space, the $\mathbf{k}$-integration can be performed via partial integration according to

$$\tilde{\Psi}_\kappa(x, t) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \left\{ \left(\frac{1}{ix} \right)^{n+1} e^{-ik_x x} \frac{\partial}{\partial k_x^n} \Psi_\kappa(k_x, t) \right\}_{k_x = 0}, \tag{28}$$

if there is only a k_x-dependence and $\tilde{\Psi}_\kappa(k_x < 0, t) = 0$. This Eq. (28) can be carried out analytically by REDUCE programming up to high orders of n. We thus obtain an analytical form of the $\tilde{\Psi}_\kappa(\mathbf{r}, t)$ permitting a consideration of boundary values such as $x \to 0$ in Eq. (28) in an exact analytical manner.

The solution $\tilde{\psi}_\kappa(\mathbf{r}, t)$ obtained will be summed up analytically or by numerical treatment, yielding the final result given by Eqs. (2) and (3).

(viii) *Initial values and growth rates.*

For the treatment of initial value problems in this method it generally must be established that the system under consideration is in its equilibrium state for time $t < t_0$. This means that

$$\psi_\mu(\mathbf{r}, t < t_0) = \psi_{\mu 0} \tag{29a}$$

and that for $t = t_0$ there exists an initial value

$$\psi_\mu(\mathbf{r}, t_0) = \varphi_\mu(\mathbf{r}). \tag{29b}$$

For the differential equations of the Navier–Stokes type treated here, Green's function has poles only in the upper ω half-plane. Thus it can be shown from Eq. (27) that

$$\tilde{\psi}_{i\kappa}(\mathbf{r}, t = t_0) = 0 \qquad (\kappa > 1), \tag{29c}$$

so that the series development Eq. (3) yields for the initial value

$$\psi_{\mu 0} + \varepsilon\tilde{\psi}_{\mu 1}(\mathbf{r}, t_0) = \varphi_\mu(\mathbf{r}). \tag{29d}$$

Hence it is possible to calculate, for a given initial value problem, the complete solution in first order and with the aid of Eq. (27) the solutions of higher order up to the desired degree. Again it is mentioned, as previously, that $\psi_{\mu 0}$ does not constitute the complete initial value, as is seen from Eq. (29d). Furthermore, the solution is independent of the choice of ε.

After a solution of the complete (non-linear) dispersion relation is obtained, non-linear growth rates for the various appearing instabilities can be calculated,

$$\gamma_\mu(\mathbf{r}, t) = Re\left(\frac{\partial}{\partial t}\ln(\psi_\mu(r, t))\right), \qquad \mu = 1,..., p. \tag{30}$$

The equations of steps (i)–(viii) then constitute the complete set of the equations of this quasi-analytical method for calculating turbulent behavior of self-confined magnetoplasmas.

3. Solution of Burgers' Equation as an Example

As a simple example of our method we have chosen Burgers' equation

$$\frac{\partial}{\partial t}u(x, t) + u(x, t)\frac{\partial}{\partial x}u(x, t) = \nu\frac{\partial^2}{\partial x^2}u(x, t), \tag{31}$$

a one-dimensional Navier–Stokes equation for incompressible flow, neglecting the influence of pressure forces. For further treatment, particular attention must be paid to the fact that with this calculation method the only problems which can be calculated are those for which a dispersion relation exists. This means that for a given initial distribution

$$u(x, t = t_0) = F(x) \tag{32}$$

the k-Fourier transform must exist. Therefore only velocity fields of the type

$$u(x, t) = u_0 + \tilde{u}(x, t), \tag{33}$$

where $u(x, t = t_0)$ is a square-integrable function in x, can be treated.

Performing now the ansatz (see Eqs. (2), (3))

$$u(x, t) = u_0 + \sum_{\kappa=1}^{M} \varepsilon^{\kappa} \tilde{u}_{\kappa}(x, t)$$

yields the nonlinear differential equation of zeroth order

$$u_0 \frac{\partial u_0}{\partial x} - \nu \frac{\partial^2 u_0}{\partial x^2} = 0 \qquad \text{(automatically fulfilled)} \tag{34a}$$

and leads to the system of linear differential equations

$$\kappa = 1: \qquad \frac{\partial}{\partial t} \tilde{u}_1 + u_0 \frac{\partial \tilde{u}_1}{\partial x} - \nu \frac{\partial^2 \tilde{u}_1}{\partial x^2} = 0, \tag{34b}$$

$$\kappa > 1: \qquad \frac{\partial}{\partial t} \tilde{u}_{\kappa} + u_0 \frac{\partial \tilde{u}_{\kappa}}{\partial x} - \nu \frac{\partial^2 \tilde{u}_k}{\partial x^2} = RS_{\kappa}(x, t), \tag{34c}$$

with

$$RS_{\kappa}(x, t) = -\sum_{j=1}^{\kappa-1} \frac{\partial \tilde{u}_j(x, t)}{\partial x} \tilde{u}_{\kappa-j}(x, t). \tag{34d}$$

The homogeneous problem $\kappa = 1$ leads to the linear dispersion relation (see Eq. (17)) by Fourier transformation

$$D(k, \omega) = i\omega - iu_0 k + \nu k^2 = 0, \tag{35}$$

resulting in

$$\omega(k) = u_0 k + i\nu k^2. \tag{36}$$

According to Eq. (20) we obtain

$$\tilde{u}_1(x, t) = \int \frac{d\omega\, dk}{4\pi^2} \hat{u}_1(k)\, e^{-i(k_x - \omega t)} \cdot 2\pi\delta(\omega - \omega(k)) \cdot \Theta(t - t_0)$$

$$= \int \frac{dk}{2\pi} \hat{u}_1(k)\, e^{-i(k_x - u_0 t)} e^{-\nu k^2 t} \cdot \Theta(t - t_0) \tag{37}$$

which shows a decaying behavior for $t \to \infty$.

To obain the solutions $\kappa > 1$ we have to calculate Green's function Eq. (11c). Performing the Fourier transformation and integration over ω-space with respect to the pole,

$$\omega(k) = u_0 k + i\nu k^2$$

in the upper half plane (see Fig. 1), there follows, after some manipulations,

$$G(x - x', t - t') = \int \frac{dk}{2\pi} e^{-ik(x - x' - u_0(t - t')) - \nu k^2 (t - t')} \Theta(t - t'). \tag{38}$$

For further treatment without loss of generality u_0 can be set equal to zero. Evaluating the integral in Eq. (38) yields

$$G(x - x', t - t') = -\frac{1}{2} \frac{\partial}{\partial x} \operatorname{erfc}\left(\frac{x - x'}{2\sqrt{\nu(t - t')}}\right) \Theta(t - t') \tag{39}$$

where

$$\operatorname{erfc}(z) = (2/\sqrt{\pi}) \int_z^\infty e^{-z'^2}\, dz', \tag{40}$$

is the complementary error function. If we assume the initial condition ($t_0 = 0$),

$$u(x, 0) = F(x), \tag{41}$$

we obtain from Eqs. (29) and (37) ($\varepsilon = 1$):

$$F(k) = \tilde{u}_1(x, 0) \leftrightarrow \hat{u}_1(k) = \int dx\, F(x)\, e^{ikx}. \tag{42}$$

Introducing the abbreviation

$$\bar{R} = \frac{1}{2\nu} \int_{-L}^{L} dx\, |F(x)|, \tag{43}$$

where $2L$ is a characteristic length of the system, we obtain with

$$f(x) = \frac{F(x)}{\int_{-L}^{L} dx\, |F(x)|} \tag{44}$$

in Eq. (37), according to (42)–(44),

$$u_1(x, t) = -\nu\bar{R}\frac{\partial}{\partial x}\int dx' f(x') \operatorname{erfc}\left(\frac{x-x'}{\sqrt{4\nu t}}\right)$$
$$= \frac{\bar{R}}{\sqrt{\pi}}\sqrt{\frac{\nu}{t}}\int dx' f(x') \exp\left(-\frac{(x-x')^2}{4\nu t}\right). \tag{45}$$

It is evident that $\tilde{u}_1(x, t)$ satisfies the diffusion equation

$$\frac{\partial}{\partial t}\tilde{u}_1(x, t) = \nu\frac{\partial^2}{\partial x^2}\tilde{u}_1(x, t). \tag{46}$$

Therefore $\tilde{u}_1(x, t)$ is the solution for weak nonlinearities ($\bar{R} \to 0$).

For higher orders of ε ($\kappa > 1$), the solution for $\tilde{u}_\kappa(x, t)$ is obtained by using Eqs. (11c), (34d), (39), (40), (43), and (44). After some manipulations we get as a result

$$\tilde{u}_\kappa(x, t) = -\frac{1}{4}\sum_{j=1}^{\kappa-1}\int dt' \frac{\partial}{\partial x}\int_\infty dx' \tilde{u}_j(x', t')\,\tilde{u}_{\kappa-j}(x', t')\frac{\partial}{\partial x'}\operatorname{erfc}\left(\frac{x-x'}{2\sqrt{\nu(t-t')}}\right). \tag{47}$$

With Eq. (47) the solutions $\tilde{u}_\kappa(x, t)$ can be evaluated by REDUCE code or appropriate integration techniques, depending on the type of the initial distribution $f(x)$. Thus we can write the complete solution of the problem

$$u(x, t) = \sum_{\kappa=1}^{\infty}\tilde{u}_\kappa(x, t) \tag{48}$$

with $\tilde{u}_1(x, t)$ from Eq. (45) and $\tilde{u}_\kappa(x, t)$, ($\kappa > 1$), from Eq. (47) provided that the series converges.

Two distinct cases of an initial distribution will be treated subsequently.

(I) For the special initial distribution

$$f(x) = \delta(x), \tag{49}$$

where the characteristic length tends to infinity ($L \to \infty$), there follows for the solution of first order,

$$\tilde{u}_1(x, t) = R\sqrt{\nu/\pi t}\, e^{-y^2/4} \tag{50}$$

with

$$y = x/\sqrt{\nu t}. \tag{51}$$

The second order solution can be derived from $\tilde{u}_1(x, t)$ with the aid of Eqs. (27) and (47). After some algebra follows

$$\tilde{u}_2(x, t) = \tfrac{1}{2}\bar{R}\tilde{u}_1(x, t)\,\mathrm{erfc}(y/2), \tag{52}$$

where $\mathrm{erfc}(x)$ is the complementary error function Eq. (40).

The method can now be continued for higher orders. It is easily seen that for nth order there follows the relation

$$\tilde{u}_n(x, t) = \tilde{u}_1(x, t)\left(-\frac{\bar{R}}{2}\,\mathrm{erfc}(y/2)\right)^n \tag{53}$$

so that the complete solution becomes

$$u(x, t) = u_1(x, t)\sum_{n=0}^{\infty}\left(\frac{\bar{R}}{2}\,\mathrm{erfc}(y/2)\right)^n \tag{54}$$

An investigation of the range of convergence of the sum in Eq. (54) shows that it converges almost everywhere [21]. The convergence is assured if

$$\frac{\bar{R}}{2}\,\mathrm{erfc}(y/2) < 1. \tag{55}$$

This means, if we define (see Eqs. (43), (51))

$$y = 2\sqrt{\Delta\bar{R}}, \qquad \Delta = \frac{x^2}{2t}, \qquad \int_{\infty} |F(x)|\,dx = 1 \tag{56}$$

that the function

$$C(R, \Delta) = \frac{\bar{R}}{2}\,\mathrm{erfc}(\sqrt{\Delta\bar{R}}) - 1 < 0. \tag{57}$$

The region of the convergence is shown in Fig. 2. It shows that there is convergence for all x, t, if $\bar{R} < 2$ and for all $\bar{R}$ if $\Delta > 0.083$. If $\bar{R} \to \infty$ convergence is given for $\Delta \neq 0$.

Therefore the sum in Eq.(54) can be expressed by a closed analytical function. There follows

$$u(x, t) = \bar{R}\sqrt{v/\pi t}\, e^{-y^2/4}/(1 + \tfrac{1}{2}\bar{R}\,\mathrm{erfc}(y/2)). \tag{58}$$

This closed form of the solution can be continued to the entire $(\Delta, \bar{R})$-space (see Fig. 2). Thus we get a solution for all x, t and $\bar{R}$.

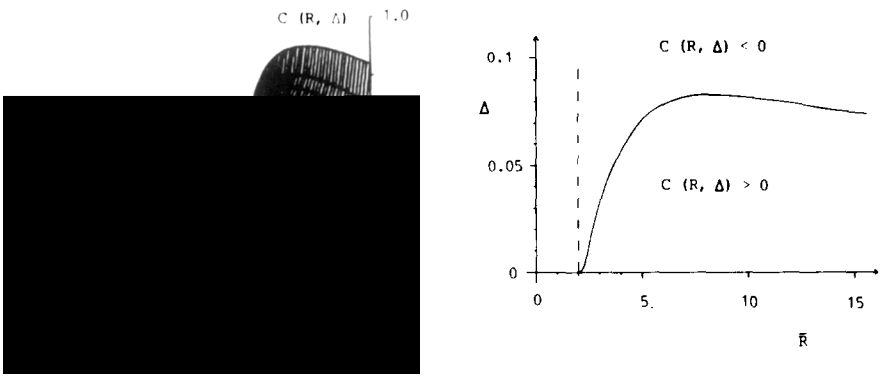

FIG. 2. Region of convergence of Eq. (54): $C(\bar{R}, \Delta) < 0$ $\Delta = x^2/(2t)$, $\int_\infty |F(x)|\, dx = 1$ (Eq. 44).

Introducing the Reynolds number as the quotient of advective force to viscous force, for which the relation

$$Re = \ln(1 + \bar{R}) \tag{59}$$

holds [21], there results, finally, for the complete solution

$$u(x, t) = \sqrt{\nu/t}\, \frac{(e^{Re} - 1)\, e^{-y^2/4}}{\sqrt{\pi} + (e^{Re} - 1) \int_{y/2}^{\infty} e^{-z^2}\, dz}. \tag{60}$$

This completely agrees with the known analytical solution of Burgers' equation with a delta-function as initial condition [18]. It shows that it is possible to calculate the exact analytical solution of this simple case of Brugers' equation with the quasi-analytical method proposed here via a transformation into Fourier space. In this simple case it was not necessary to use FORTRAN for any part of the operation.

To show the correctness of this FORTRAN part of the hybrid code, the same problem was again calculated in the manner used for more complicated systems, namely by implementing the REDUCE manipulations with numerical calculations.

Figure 3 shows the exact solution of Burgers' equation (60) for a Reynolds number of 1000. Also indicated in this figure is an approximate solution with the method described in this report. However, contrary to the general practice of continuing approximations until a maximum error of 1.5% is obtained, the solution in Fig. 3 has a maximum error of 6.5% with 70 members of the series development, instead of 292 for an accuracy of 1.5 percent.

(II) As a second example, the initial distribution

$$F(x) = U \sin(kx) \tag{61}$$

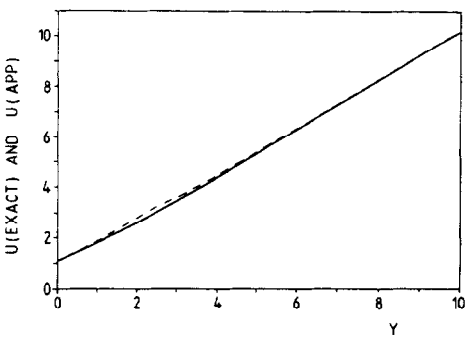

FIG. 3. Exact solution (—) of Burgers' equation compared with the solution containing 70 members of series development (---) for a Reynolds number of 1000.

was chosen. According to Eqs. (43) and (44) we get as the "Reynolds"-number

$$\bar{R} = \frac{U}{\nu k} \tag{62}$$

with the characteristic length $2L = \pi/k$. Hence,

$$f(x) = \tfrac{1}{2}k \sin(kx), \tag{63}$$

yielding, with the aid of Eq. (45),

$$u_1(x) = U \sin(kx)\, e^{-\nu k^2 t}. \tag{64}$$

According to Eq. (47), there follows for the solution of second order

$$u_2(x) = \tfrac{1}{4} U\bar{R}(e^{-2\nu k^2 t} - 1)\, e^{-2\nu k^2 t} \sin(2kx). \tag{65}$$

The solutions of higher order ($\kappa > 2$) can be evaluated successively using Eq. (47), however, for practical treatment it is more practical to perform further calculations using the REDUCE–FORTRAN-hybrid code. The solution of the problem is plotted in Fig. 4, which shows the typical behavior of the solutions of Burgers' equation. It is interesting to note that the growth rate equation (30) at $t = 0$,

$$\gamma(x, t = 0) = \frac{\dot{u}_1(x, 0) + \dot{u}_2(x, 0)}{u_1(x, 0)} = \nu k^2(1 + \bar{R} \cos kx)$$

obviously shows the dependence on the "Reynolds"-number $\bar{R}$.

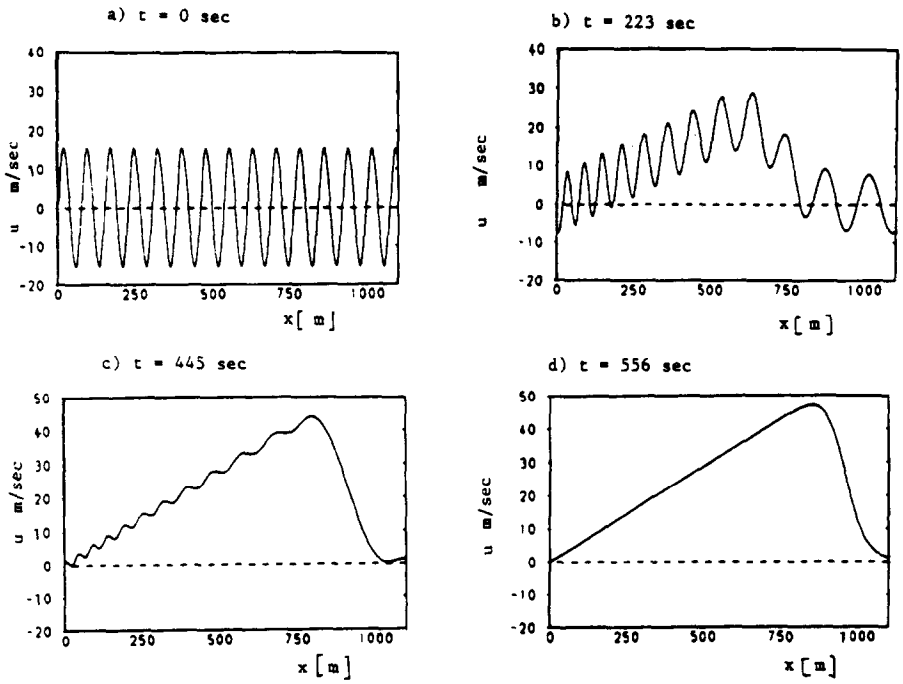

FIG. 4. Solution of Burgers' equation for the initial distribution $F(x) = U \sin kx$. $U = 15.45$ m/sec, $k = 0.082$ 1/m, $\bar{R} = 3.5 \cdot 10^5$.

4. CONCLUSIONS

In this paper we presented a quasi-analytical method for solving systems of non-linear differential equations of the Navier–Stokes type including strong turbulence with particular reference to self-confined system. The method is characterized by a transformation into Fourier space, yielding a macroscopic dispersion relation which permits the study of the dispersive behavior of the physical system. Growth rates and spectra of various instabilitites can thus be obtained. Complex ω-integration and ensuing **k**-integration was carried out resulting in a description of the complete behavior in coordinate space and time. The analytical manipulation is achieved by REDUCE, necessary numerical solution of the dispersion relation and fit programs are made with FORTRAN. The fact that also the back-transformation into $(\mathbf{r}, t)$-space is performed analytically avoids problems associated with steep gradients. This REDUCE–FORTRAN hybrid code was employed in solving problems of plasma focus dynamics [9, 19, 20]. The reason for selecting Burgers' equation for the demonstration in section 3 of this paper was the fact that for one it is a non-linear equation of the Navier–Stokes type and, second, there exists a closed analytical solution for a specific initial distribution. In this way it was possible not

only to demonstrate the application of the REDUCE procedure in our hybrid code but also to show that if there exists a closed analytical solution obtained directly, this solution is also obtained by our mathematical formalism via a transformation into Fourier space and back. We consider this a proof that our method is in principle correct, though this does certainly not prove that it will be applicable in all cases of flow dynamics. As pointed out already, it will be necessary to consider the convergence behavior in each individual case. Furthermore, the boundary value problem treated in this paper permits only the solution for self-confined systems. An extension for bounded inhomogeneous systems is being developed.

The advantage of this hybrid code lies in the possibility to treat complex systems of equations, such as they were established for a four-fluid theory [10], with a reasonable expenditure of calculation time. It is again pointed out that this reasonable time expenditure is only achieved through the combination of analytical and numerical procedures including fit programs. A pure REDUCE code would, for complicated cases, rapidly exhaust the central memory capacity of available computer systems, while in treating turbulence phenomena by purely numerical methods the accuracy would rapidly decrease [1].

In this hybrid code the dispersion relation is established by REDUCE, the solution of the dispersion relation is mostly achieved by numerical (FORTRAN) procedure, starting with the homogeneous solution. The numerical results then are fitted to an analytical function which then is used in the higher order solution. This procedure is continued in an alternation of REDUCE procedure, numerical calculations and fit programming. Especially in the REDUCE part for the solution of higher order members it is possible to repeatedly use the same analytical formula, so that no new REDUCE time is required. Thus the fit programming helps to reduce not only computer time but also storage space, a fact which was especially noticeable in the four-fluid calculations of plasma dynamics.

To give an approximate idea of the time expenditure for this REDUCE–FORTRAN hybrid code, Table I shows the computer time required for the solution of Burgers' equation as a function of Reynolds number. As the time expenditure is dependent on the degree of nonlinearity, the Reynolds number is the decisive parameter.

The times for REDUCE are required only once for solving a specific problem, while the FORTRAN times will be required for each example with a new set of boundary and/or initial conditions.

It is of course evident that in this developed quasi-analytical method with the REDUCE–FORTRAN hybrid code, no general rules for the treatment of nonlinear problems can be given. Each individual case must be treated separately, as also the convergence of the series Eq. (3) must be investigated for each individual case.

It should be of interest to note that A. Bers and collaborators have employed symbolic computation in the treatment of nonlinear wave interactions [22]. Their method has many similarities to the quasi-analytical method discussed here, especially the transformation into Fourier space. The difference lies in the fact that Bers *et al.* use a true perturbation development which is broken off after a limited

TABLE I

Order of Approximation as a Function of Reynolds Number for a Maximum Deviation of 1.5% from the Exact Solution

Renumber	Number of approximation	VAX 750 REDUCE time (sec)	CYBER 175 FORTRAN time (sec)
1	72	720	72.5
5	87	792	89
10	98	828	97
50	140	924	118
100	165	960	129
500	246	1068	158
1000	292	1104	172
5000	437	1224	210
10000	520	1284	229

number of members. The symbolic code used by these authors was MACSYMA. Gladd [2] also used MACSYMA in describing symbolic manipulation techniques for plasma kinetic theory. He also argues that a combination of symbolic and numerical procedures are required, however, he does not give any details in this respect.

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REFERENCES

1. P. G. SAFFMANN, in *Structure and Mechanism of Turbulence*—II, *Proc. Berlin* 1977 (Springer, Berlin, 1977), p. 273.
2. A. C. HEARN, *REDUCE 2–Users Manual*, Report UCP-19 (Univ. of Utah, Salt Lake City, 1973).
3. A. C. HEARN, *Standard LISP*, Stanford Artificial Intelligence Project Mans. AI-90, 1969 (unpublished).
4. H. STOYAN, *LISP Programmierbuch* (Akademie-Verlag, Berlin, 1978).
5. A. HAYD AND P. MEINKE, *Turbulenzstabilisierte Formen und ihre Bedeutung für die Plasmaphysik*, M.A.N. Report, B 090 032-EDS-001, Munich, 1980 (unpublished).
6. H. J. KAEPPELER, *Instabilities in the Plasma Focus*, Report IPF-75-1 (Institut für Plasmaforschung, Stuttgart, 1975).
7. H. E. WILHELM, *Beitr. Plasmaphys.* **3** (1, 2), 25 (1963).
8. A. HAYD, H. J. KAEPPELER, AND P. MEINKE, Report IPF-82-1 (Institut für Plasmaforschung, Stuttgart, 1982).
9. A. HAYD, H. J. KAEPPELER, M. MAURER, AND P. MEINKE, Report IPF-82-7 (Institut für Plasmaforschung, Stuttgart, 1982); M.A.N., Report B 099 042-EDS-020 Munich, 1982, [German].
10. H. J. KAEPPELER, Report IPF-83-3 (Institut für Plasmaforschung, Stuttgart, 1982).

11. L. V. KANTOROVICH AND V. I. KRYLOV, *Approximate Methods in Higher Analysis* (Interscience, New York, 1985); B. A. FINLAYSON AND L. E. SCRIVEN, *Appl. Mech. Rev.* **19**, 735 (1966).
12. F. CAP, *Handbook on Plasma Instabilities*, Vol. 1 (Academic Press, New York, 1976).
13. E. RÄUCHLE, Report IPF-72-8 (Institut für Plasmaforschung, Stuttgart, 1979).
14. A. L. MC CARTHY AND E. RÄUCHLE, Report IPF-80-3 (Institut für Plasmaforschung, Stuttgart, 1980).
15. F. MOSER, Report IPF-74-2 (Institut für Plasmaforschung, Stuttgart, 1974).
16. F. MOSER, *Plasma Phys.* **17**, 821 (1975).
17. L. LANDAU, *J. Phys. USSR* **10**, 25 (1946).
18. G. B. WHITHAM, *Linear and Nonlinear Waves* (Wiley-Interscience, New York, 1974).
19. A. HAYD, M. MAURER, P. MEINKE, AND H. J. KAEPPELER, M.A.N. Report B 099 042-ED-003, Munich, 1983 (unpublished).
20. H. J. KAEPPELER, A. HAYD, M. MAURER, AND P. MEINKE, Report IPF-83-2 (Institut für Plasmaforschung, Stuttgart, 1983).
21. M. MAURER AND A. HAYD, M.A.N. Report ED-046, Munich, 1984 (unpublished).
22. A. BERS, J. L. KULP AND C. F. F. KARNEY, *Comput. Comm.* **12**, 81 (1976).
23. N. T. GLADD, *J. Comput. Phys.* **56**, 175 (1984).